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ON THE HILBERT FUNCTION OF THE UNION OF TWO LINEAR STAR-CONFIGURATIONS IN $\mathbb{P}^2$

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Abstract. It has been proved that the union of two linear star-configura- tions in $\mathbb{P}^2$ of type $t \times s$ for $3 \leq t \leq 9$ and $3 \leq t \leq s$ has generic Hilbert function. We extend the condition to $t = 10$, so that it is true for $3 \leq t \leq 10$, which generalizes the result of [7].

1. Introduction

In recent years Hilbert functions and minimal free resolutions of star-confi- gurations have been extensively studied (see [1, 7, 8]) but those of fat star-configurations are in a stage of exploring. In this paper, we discuss the Hilbert function of the union of two linear star-configurations in the 2 dimensional projective space $\mathbb{P}^2$ over an algebraically closed field k of an arbitrary characteristic. Let $R = k[x_0, x_1, \ldots, x_n]$ be an $(n+1)$-variable polynomial ring over a field k and let I be a homogeneous ideal of R (or the ideal of a subscheme in $\mathbb{P}^n$). Then the numerical function

$$\mathbf{H}(R/I, t) := \dim_k R_t - \dim_k I_t$$

is called a *Hilbert function* of the ring R/I. If $I := I_{\mathbb{X}}$ is the ideal of a subscheme $\mathbb{X}$ in $\mathbb{P}^n$, then we denote the Hilbert function of $\mathbb{X}$ by

$$\mathbf{H}(R/I_{\mathbb{X}}, t) := \mathbf{H}_{\mathbb{X}}(t).$$

It has been also in question to find the vector space dimension of the graded components of coordinate rings of subschemes $\mathbb{X}$ in $\mathbb{P}^n$, that is, the Hilbert function of $\mathbb{X}$ (see [2, 3, 4, 5, 6]).

To introduce fat star-configuration and star-configuration, we start with a variety of some specific ideal of R. In [1], the authors proved that if

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$F_1, F_2, \dots,$
F_r are general forms in R with $r \geq 2$, then

$$\bigcap_{1\leq i<j\leq r}(F_i^a, F_j^b) = (\tilde{F}_1, \cdots, \tilde{F}_r),$$

where

$$\tilde{F}_j = \frac{\prod_{i=1}^r F_i^a}{F_j^a} \cdot (F_{j+1}\cdots F_r)^{b-a} \quad \text{for} \quad j = 1, \dots, r,\ 1 \leq a \leq b.$$

The variety $\mathbb{X}$ in $\mathbb{P}^n$ of the ideal

$$(\tilde{F}_1, \dots, \tilde{F}_r) = \bigcap_{1\leq i<j\leq r}(F_i^a, F_j^b)$$

is called a *fat star-configuration* in $\mathbb{P}^n$ of type (r, a, b) defined by general forms $F_1 \dots, F_r$. In particular, if $a = b = 1$ then we simply call $\mathbb{X}$ a *star-configuration* in $\mathbb{P}^n$ of type r defined by general forms $F_1 \dots, F_r$. Furthermore, if $F_1, \dots, F_r$ are all general *linear* forms, then $\mathbb{X}$ is called a *fat linear star-configuration* of type (r, a, b). If $\mathbb{X}$ is a fat linear star-configuration of type $(r, 1, 1)$, then we simply call $\mathbb{X}$ a linear star-configuration of type r.

In [8], the author studied the relation between the dimension of secant varieties $\mathrm{Sec}_r(\mathrm{Split}_d(\mathbb{P}^n))$ to the variety of completely decomposable forms in R and the Hilbert function of the union $\mathbb{X}$ of two linear star-configurations in $\mathbb{P}^2$ of type $t \times s$, and showed that

1. $\mathbb{X}$ has generic Hilbert function for $3 \leq t \leq 9$ and $3 \leq t \leq s$, and
2. the secant variety $\mathrm{Sec}_1(\mathrm{Split}_d(\mathbb{P}^n))$ to the variety of completely decomposable forms in R is not defective for $d \geq 3$.

We therefore focus on extension of the result (1) in Section 2 and prove that the Hilbert function of the union $\mathbb{X}$ of two linear star-configurations in $\mathbb{P}^2$ of type $t \times s$, and that $\mathbb{X}$ has generic Hilbert function when $3 \leq t \leq 10$ and $3 \leq t \leq s$ (see Corollary 2.3). More precisely, we use lines vanishing on multiple points to apply Bezóut's theorem. This is a different idea from [7], which allows us to expand the result of [7].

2. The Hilbert function of the union of star-configurations in $\mathbb{P}^2$

Let $\mathbb{X} := \mathbb{X}^{(t,s)}$ be the union of two linear star-configurations $\mathbb{X}_1$ and $\mathbb{X}_2$ in $\mathbb{P}^2$ of types t and s (type $t \times s$ for short), defined by general linear forms $M_1, \dots, M_t$ and $L_1, L_2, \dots, L_s$ for $3 \leq t \leq s$, respectively, and let $\mathbb{Y} := \mathbb{X}^{(t,s-1)}$ be the union of two linear star-configurations $\mathbb{Y}_1 = \mathbb{X}_1$ and $\mathbb{Y}_2$ of type $t \times (s-1)$ defined by linear forms $M_1, \dots, M_t$ and $L_2, \dots, L_s$, respectively. Note that $\mathbb{Y}_2 \subseteq \mathbb{X}_2$ and $\mathbb{Y} \subseteq \mathbb{X}$.

The linear forms create points and lines in $\mathbb{P}^2$: $Q_{i,j}$ is a point in $\mathbb{X}_1$ defined by linear forms M_i and M_j; $P_{i,j}$ is a point in $\mathbb{X}_2$ defined by linear forms L_i and L_j with $i < j$; $\mathbb{L}_i$ and $\mathbb{M}_j$ are lines defined by linear forms L_i and M_j for $i = 1, ..., s$ and $j = 1, ..., 10$, respectively. We define $G := L_2 \cdots L_s$, a product of $(s-1)$ linear forms, and $L := L_1$.

In this section, we shall prove that the Hilbert function of the union of two linear star-configurations in $\mathbb{P}^2$ of types 10 and s with $s \geq 10$ has generic Hilbert function. The proof is based on mainly two ideas. The first idea is that if $\mathbb{X}'$ is the union of two finite sets of points defined by linear forms $M_1, \ldots, M_t$ and $L_1, L_2, \ldots, L_s$ in R (not necessarily general), respectively, then the points in $\mathbb{X}$ are more general than the points in $\mathbb{X}'$. This implies for every $i \geq 0$ we get

$$\mathbf{H}_{\mathbb{X}'}(i) \leq \mathbf{H}_{\mathbb{X}}(i).$$

The second idea is *Bezout*'s Theorem in $\mathbb{P}^2$ to find the union $\mathbb{X}'$ of two sets of points defined by linear forms $M_1, \ldots, M_t$ and $L_1, L_2, \ldots, L_s$ in R, respectively, such that

$$\begin{aligned} \mathbf{H}_{\mathbb{X}}(i) &= \mathbf{H}_{\mathbb{X}'}(i) \\ &= \min\left\{|\mathbb{X}|, \binom{i+2}{2}\right\} \quad \text{for some} \quad i \geq 0. \end{aligned}$$

In other words, if a form F of degree d in R vanishes on $(d+1)$-points on the line defined by a linear form M in R, then F is divided by the linear form M. Throughout this section, we shall not distinguish $\mathbb{X}$ from $\mathbb{X}'$ for convenience. For the rest of this section, we shall often use the following exact sequence.

(2.1)

$$0 \to R/I_{\mathbb{X}} \to R/I_{\mathbb{Y}} \oplus R/(L,G) \to R/(I_{\mathbb{Y}}, L, G) \to 0.$$

Proposition 2.1. $\mathbb{X} := \mathbb{X}^{(10,10)}$ *has generic Hilbert function.*

Proof. By [7, Theorem 3.8] and equation (2.1),

$$\begin{array}{rcccccccccc} & & & & & & & \text{12-th} & & \\ \mathbf{H}(R/I_{\mathbb{X}}, -) & : & 1 & 3 & \cdots & 66 & 78 & \beta & 90 & \to, \\ \mathbf{H}(R/I_{\mathbb{Y}}, -) & : & 1 & 3 & \cdots & 66 & 78 & 81 & 81 & \to, \\ \mathbf{H}(R/(L,G), -) & : & 1 & 2 & \cdots & 9 & 9 & 9 & 9 & \to, \\ \mathbf{H}(R/(I_{\mathbb{Y}}, L, G), -) & : & 1 & 2 & \cdots & 9 & 9 & \gamma & 0 & \to, \\ \mathbf{H}(R/(I_{\mathbb{Y}}, L), -) & : & 1 & 2 & \cdots & 11 & 12 & 3 & 0 & \to, \end{array}$$

where β and γ are unknown. Hence it suffices to show that $\beta = 90$, that is, $\dim_k(I_{\mathbb{X}})_{12} = 1$.

Let $\mathbb{X}_1$ and $\mathbb{X}_2$ be finite sets of 45 points defined by $L_1, L_2, L_3, L_4, L_5, L_6, L_7, L_8, L_9, L_{10}$, and $M_1, M_2, M_3, M_4, M_5, M_6, M_7, M_8, M_9, M_{10}$, respectively, where L_i and M_j are linear forms (see Figure 1), and $\mathbb{X} := \mathbb{X}_1 \cup \mathbb{X}_2$.

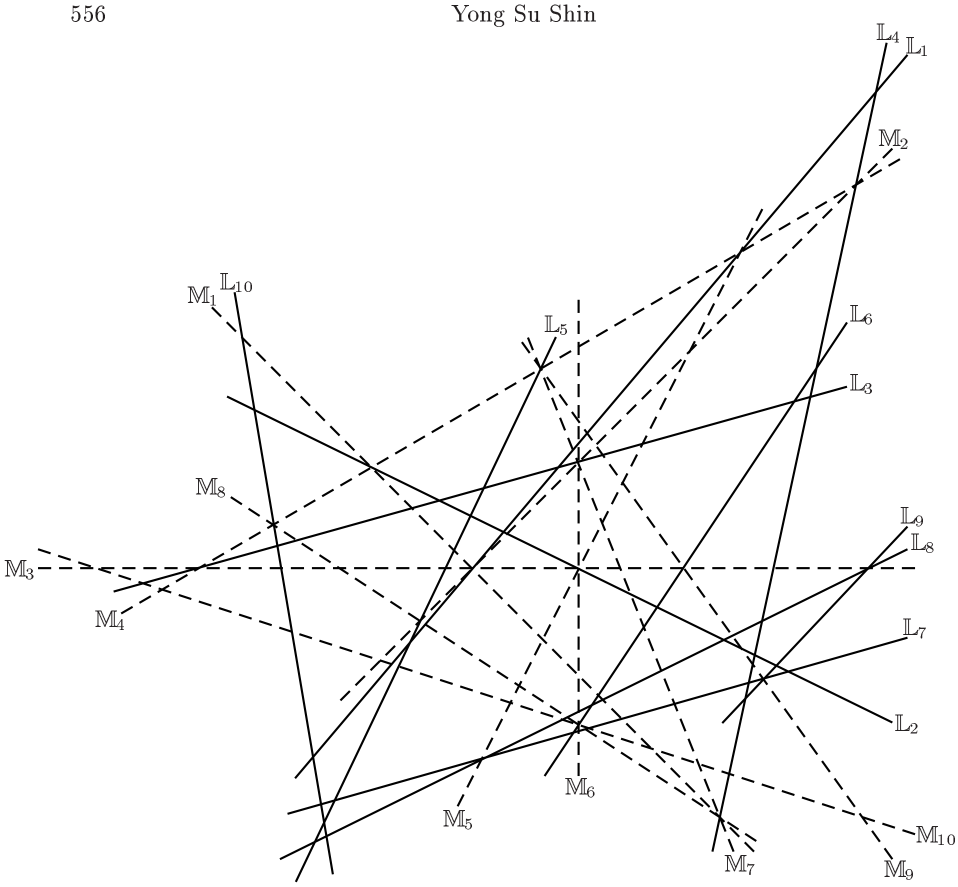

FIGURE 1

As shown in Figure 1, we have that

L_1	vanishes on 13 points	$P_{1,2}, P_{1,3}, P_{1,4}, P_{1,5}, P_{1,6}, P_{1,7}, P_{1,8}, P_{1,9}, P_{1,10}, Q_{1,2}, Q_{1,3}, Q_{2,3}, Q_{4,5},$
L_2	vanishes on 12 points	$P_{2,3}, P_{2,4}, P_{2,5}, P_{2,6}, P_{2,7}, P_{2,8}, P_{2,9}, P_{2,10}, Q_{1,4}, Q_{3,5}, Q_{3,6}, Q_{5,6},$
L_3	vanishes on 11 points	$P_{3,4}, P_{3,5}, P_{3,6}, P_{3,7}, P_{3,8}, P_{3,9}, P_{3,10}, Q_{2,6}, Q_{2,7}, Q_{3,4}, Q_{6,7},$
L_4	vanishes on 10 points	$P_{4,5}, P_{4,6}, P_{4,7}, P_{4,8}, P_{4,9}, P_{4,10}, Q_{1,7}, Q_{1,8}, Q_{2,4}, Q_{7,8},$
L_5	vanishes on 9 points	$P_{5,6}, P_{5,7}, P_{5,8}, P_{5,9}, P_{5,10}, Q_{2,8}, Q_{4,7}, Q_{4,9}, Q_{7,9},$
L_6	vanishes on 8 points	$P_{6,7}, P_{6,8}, P_{6,9}, P_{6,10}, Q_{3,9}, Q_{6,8}, Q_{6,10}, Q_{8,10},$
M_{10}	vanishes on 7 points	$Q_{1,10}, Q_{2,10}, Q_{3,10}, Q_{4,10}, Q_{5,10}, Q_{7,10}, Q_{9,10},$
M_5	vanishes on 6 points	$P_{7,8}, Q_{1,5}, Q_{2,5}, Q_{5,7}, Q_{5,8}, Q_{5,9},$
M_9	vanishes on 5 points	$P_{7,9}, Q_{1,9}, Q_{2,9}, Q_{6,9}, Q_{8,9},$

L_{10}	vanishes on 4 points	$P_{7,10}, P_{8,10}, P_{9,10}, Q_{4,8}$,
M_3	vanishes on 3 points	$P_{8,9}, Q_{3,7}, Q_{3,8}$,
M_6	vanishes on 2 points	$Q_{1,6}, Q_{4,6}$.

By Bezout's theorem, for every $N \in (I_{\mathbb{X}})_{12}$,

$$N = \alpha L_1 \cdots L_6 L_{10} M_3 M_5 M_6 M_9 M_{10}$$

for some $\alpha \in k$. Thus

$$\dim_k (I_{\mathbb{X}})_{12} = 1,$$

which completes the proof. □

THEOREM 2.2. $\mathbb{X} := \mathbb{X}^{(10,s)}$ *has generic Hilbert function for* $s \geq 10$.

Proof. By Proposition 2.1, this theorem holds for $s = 10$. Now assume $s \geq 11$. We shall prove these with 5 cases.

Case 1. Let $11 \leq s \leq 14$. First assume $s = 11$. By Proposition 2.1,

							12-th		
$\mathbf{H}(R/I_{\mathbb{X}}, -)$	:	1	3	$\cdots$	66	78	β	100	$\rightarrow$,
$\mathbf{H}(R/I_{\mathbb{Y}}, -)$	:	1	3	$\cdots$	66	78	90	90	$\rightarrow$,
$\mathbf{H}(R/(L,G), -)$	:	1	2	$\cdots$	10	10	10	10	$\rightarrow$,
$\mathbf{H}(R/(I_{\mathbb{Y}}, L, G), -)$	:	1	2	$\cdots$	10	10	γ	0	$\rightarrow$,
$\mathbf{H}(R/(I_{\mathbb{Y}}, L), -)$	:	1	2	$\cdots$	11	12	12	0	$\rightarrow$,

where β and γ are unknown. Hence it suffices to show that $\beta = 91$, that is, $\dim_k (I_{\mathbb{X}})_{12} = 0$.

As shown in Figure 2, we have that

L_1	vanishes on 13 points	$P_{1,2}, P_{1,3}, P_{1,4}, P_{1,5}, P_{1,6}, P_{1,7}, P_{1,8}, P_{1,9}$, $P_{1,10}, P_{1,11}, Q_{1,2}, Q_{1,3}, Q_{2,3}$,
L_2	vanishes on 12 points	$P_{2,3}, P_{2,4}, P_{2,5}, P_{2,6}, P_{2,7}, P_{2,8}, P_{2,9}, P_{2,10}$, $P_{2,11}, Q_{3,4}, Q_{3,5}, Q_{4,5}$,
L_3	vanishes on 11 points	$P_{3,4}, P_{3,5}, P_{3,6}, P_{3,7}, P_{3,8}, P_{3,9}, P_{3,10}, P_{3,11}$, $P_{3,11}, Q_{5,6}, Q_{5,7}, Q_{6,7}$,
L_4	vanishes on 10 points	$P_{4,5}, P_{4,6}, P_{4,7}, P_{4,8}, P_{4,9}, P_{4,10}, P_{4,11}, Q_{1,5}$, $Q_{1,8}, Q_{5,8}$,
M_{10}	vanishes on 9 points	$Q_{1,10}, Q_{2,10}, Q_{3,10}, Q_{4,10}, Q_{5,10}, Q_{6,10}, Q_{7,10}$, $Q_{8,10}, Q_{9,10}$,

M_9	vanishes on 8 points	$Q_{1,9}, Q_{2,9}, Q_{3,9}, Q_{4,9}, Q_{5,9}, Q_{6,9}, Q_{7,9}, Q_{8,9}$,
L_5	vanishes on 7 points	$P_{5,6}, P_{5,7}, P_{5,8}, P_{5,9}, P_{5,10}, P_{5,11}, Q_{1,6}$,
L_6	vanishes on 6 points	$P_{6,7}, P_{6,8}, P_{6,9}, P_{6,10}, P_{6,11}, Q_{1,7}$,
M_8	vanishes on 5 points	$Q_{2,8}, Q_{3,8}, Q_{4,8}, Q_{6,8}, Q_{7,8}$,
L_7	vanishes on 4 points	$P_{7,8}, P_{7,9}, P_{7,10}, P_{7,11}$,
L_8	vanishes on 3 points	$P_{8,9}, P_{8,10}, P_{8,11}$,
L_9	vanishes on 2 points	$P_{9,10}, P_{9,11}$.

By Bezout's theorem, for every $N \in (I_{\mathbb{X}})_{12}$,

$$N = \alpha L_1 \cdots L_9 M_8 M_9 M_{10}$$

for some $\alpha \in k$. Moreover, since all the 10-points $P_{10,11}, Q_{1,4}, Q_{2,4}, Q_{2,5}, Q_{2,6}, Q_{2,7}, Q_{3,6}, Q_{3,7}, Q_{4,6}$, and $Q_{4,7}$ are not on a single line, and N has to vanish on those 10-points, where none of $L_1, \ldots, L_9, M_8, M_9, M_{10}$ can vanish, we see that $N = 0$. Therefore,

$$(I_{\mathbb{X}})_{12} = \{0\},$$

as we wished.

By the same idea as in the case for $s = 11$, one can show that $\mathbb{X}^{(10,s)}$ has generic Hilbert function for $s = 12, 13$, and 14 as well. This completes the proof for *Case* 1.

Case 2. Let $s = 15$. It is from $s = 14$ that

							15-th		
$\mathbf{H}(R/I_{\mathbb{X}}, -)$	:	1	3	$\cdots$	105	120	β	150	$\rightarrow$,
$\mathbf{H}(R/I_{\mathbb{Y}}, -)$	:	1	3	$\cdots$	105	120	136	136	$\rightarrow$,
$\mathbf{H}(R/(L,G), -)$	:	1	2	$\cdots$	14	14	14	14	$\rightarrow$,
$\mathbf{H}(R/(I_{\mathbb{Y}}, L, G), -)$	:	1	2	$\cdots$	14	14	γ	0	$\rightarrow$,
$\mathbf{H}(R/(I_{\mathbb{Y}}, L), -)$	:	1	2	$\cdots$	14	15	14	0	$\rightarrow$,

where β and γ are unknown. Since $(I_{\mathbb{X}})_{15} \subseteq (I_{\mathbb{Y}})_{15} = \{0\}$, we see that $\mathbb{X}^{(10,15)}$ has generic Hilbert function, as we wanted.

Case 3. Let $16 \le s \le 23$. We can show that $\mathbb{X}^{(10,s)}$ has generic Hilbert function by the same ideas as in *Case* 1 for $16 \le s \le 22$ and as in *Case* 2 for $s = 23$.

Case 4. Let $24 \le s \le 45$. We first prove for the case $s = 24$ that $\mathbb{X}^{(10,24)}$ has generic Hilbert function. The rest of the case for $25 \le s \le 45$ can be also proved by the same methods and thus omitted.

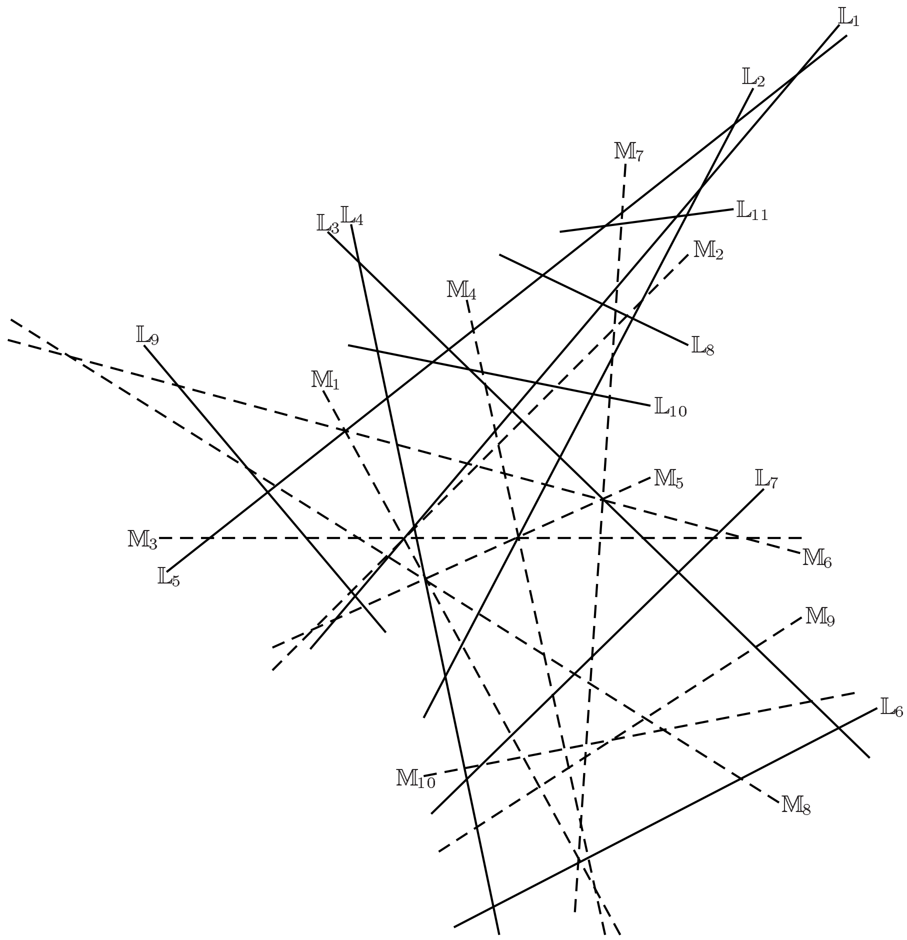

FIGURE 2

From the case for $s = 23$, we have

							23-rd		
$\mathbf{H}(R/I_{\mathbb{X}}, -)$	:	1	3	$\cdots$	253	276	β	321	$\rightarrow,$
$\mathbf{H}(R/I_{\mathbb{Y}}, -)$	:	1	3	$\cdots$	253	276	298	298	$\rightarrow,$
$\mathbf{H}(R/(L, G), -)$	:	1	2	$\cdots$	23	23	23	23	$\rightarrow,$
$\mathbf{H}(R/(I_{\mathbb{Y}}, L, G), -)$	:	1	2	$\cdots$	23	23	γ	0	$\rightarrow,$
$\mathbf{H}(R/(I_{\mathbb{Y}}, L), -)$	:	1	2	$\cdots$	23	23	22	0	$\rightarrow,$

where β and γ are unknown. Hence it suffices to show that $\beta = 300$, that is, $(I_{\mathbb{X}})_{23} = \{0\}$.

Let $\mathcal{P}_i = \{P_{i,i+1}, \dots, P_{i,24}\}$ for $i = 1, \dots, 23$. As done for the previous case, we can make $\mathbb{X}$ satisfy the following:

$$\begin{array}{lll}
L_1 & \text{vanishes on 24 points} & \mathcal{P}_1 \cup \{Q_{1,2}\}, \\
 & \vdots & \\
L_9 & \text{vanishes on 16 points} & \mathcal{P}_9 \cup \{Q_{1,10}\}, \\
L_{10} & \text{vanishes on 15 points} & \mathcal{P}_{10} \cup \{Q_{2,3}\}, \\
 & \vdots & \\
L_{17} & \text{vanishes on 8 points} & \mathcal{P}_{17} \cup \{Q_{2,10}\}, \\
L_{18} & \text{vanishes on 7 points} & \mathcal{P}_{18} \cup \{Q_{3,4}\}, \\
 & \vdots & \\
L_{23} & \text{vanishes on 2 points} & \mathcal{P}_{23} \cup \{Q_{3,9}\},
\end{array}$$

By Bezout's theorem, for every $N \in (I_{\mathbb{X}})_{23}$,

$$N = \alpha L_1 \cdots L_{23}$$

for some $\alpha \in k$. Moreover, since N has to vanish on 22 points in $\mathbb{X}_2 - \{Q_{1,2}, \dots, Q_{1,10}, Q_{2,3}, \cdots, Q_{2,10}, Q_{3,4}, \dots, Q_{3,9}\}$, where none of $L_1, \dots, L_{23}$, vanishes, we see that $N = 0$. Therefore, $(I_{\mathbb{X}})_{23} = \{0\}$, as we wished.

Case 5. Let $s \geq 46$. By induction on s, $R/I_{\mathbb{Y}}$ has generic Hilbert function and thus we have

$$\binom{(s-3)+2}{2} + \binom{10}{2} \leq \binom{(s-3)+2}{2} + (s-1) = \binom{(s-2)+2}{2},$$

and

$$\begin{aligned}
\mathbf{H}(R/I_{\mathbb{Y}}, s-2) &= \min\left\{\binom{(s-3)+2}{2} + \binom{10}{2}, \binom{(s-2)+2}{2}\right\} \\
&= \binom{(s-3)+2}{2} + \binom{10}{2}.
\end{aligned}$$

Hence

$$\begin{array}{rclcccc}
 & & & & (s-2)\text{-nd} & & \\
\mathbf{H}(R/I_{\mathbb{X}}, -) & : & 1 & \cdots & \beta & \binom{(s-2)+2}{2} + \binom{10}{2} & \to, \\
\mathbf{H}(R/I_{\mathbb{Y}}, -) & : & 1 & \cdots & \binom{(s-3)+2}{2} + \binom{10}{2} & \binom{(s-3)+2}{2} + \binom{10}{2} & \to, \\
\mathbf{H}(R/(L,G), -) & : & 1 & \cdots & s-1 & s-1 & \to, \\
\mathbf{H}(R/(I_{\mathbb{Y}}, L, G), -) & : & 1 & \cdots & \gamma & 0 & \to, \\
\mathbf{H}(R/(I_{\mathbb{Y}}, L), -) & : & 1 & \cdots & \binom{10}{2} & 0 & \to,
\end{array}$$

where β and γ are unknown.

Moreover, since $\deg G = s-1$, we have

$$\begin{array}{rcl}\mathbf{H}(R/(I_{\mathbb{Y}},L,G),s-2) & = & \mathbf{H}(R/(I_{\mathbb{Y}},L),s-2) \\ & = & \binom{10}{2},\end{array}$$

and thus

$$\begin{array}{rl} & \mathbf{H}(R/I_{\mathbb{X}},s-2) \\ = & \mathbf{H}(R/I_{\mathbb{Y}},s-2)+\mathbf{H}(R/(L,G),s-2)-\mathbf{H}(R/(I_{\mathbb{Y}},L,G),s-2) \\ = & \binom{(s-3)+2}{2}+\binom{10}{2}+(s-1)-\binom{10}{2} \\ = & \binom{(s-3)+2}{2}+(s-1) \\ = & \binom{(s-2)+2}{2}.\end{array}$$

This means that $\mathbb{X}^{(10,s)}$ has generic Hilbert function, which proves *Case 5*, and thus completes the proof of the theorem. □

If we couple the work done in [7] with Theorem 2.2, we obtain the following corollary.

COROLLARY 2.3. *If $\mathbb{X}$ is the union of two linear star-configurations in $\mathbb{P}^2$ of type $t \times s$ with $3 \le t \le 10$ and $3 \le t \le s$, then $\mathbb{X}$ has generic Hilbert function.*

By Corollary 2.3, [7, Question 5.6] can be revised as follows.

QUESTION 2.4 ([7, Question 5.6 (revised)]). Let $\mathbb{X} := \mathbb{X}^{(t,s)}$ be the union of two linear star-configurations in $\mathbb{P}^2$ of type $t\times s$ with $11 \le t \le s$. Does $R/I_{\mathbb{X}}$ have generic Hilbert function?

REMARK 2.5. Indeed, we tried to answer to Question 2.4, and found the affirmative answer for $t = 11$ as well. But we do not introduce the proof for $t = 11$ in this paper because the ideas are basically the same as for $t = 10$. It seems likely that we can prove Question 2.4 case by case on t, but a general proof is not yet available.

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